Principal type inference with subtyping

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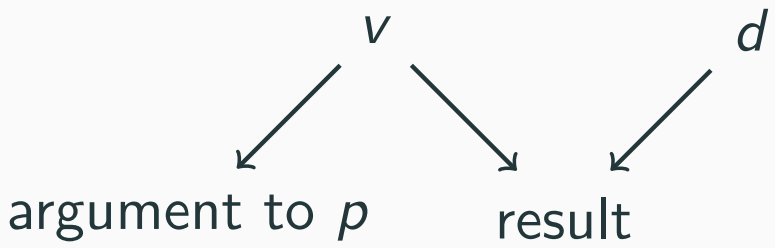

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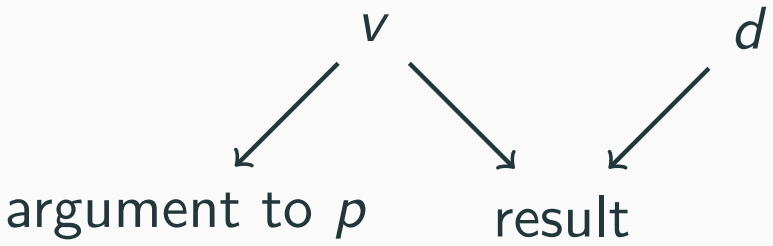

$$\mathit{MLsub} : \forall\alpha.(\alpha \to \mathsf{bool}) \to \alpha \to \beta \to (\alpha \sqcup \beta)$$

Defining types

Extensibility

When I add more types,

no programs should break.

Constructing types syntactically

Types:

$$\tau ::= \mathsf{bool} \mid \tau_1 \to \tau_2 \mid \{\ell_1 : \tau_1, \ldots, \ell_n : \tau_n\}$$

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Types with subtyping:

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Types with subtyping, least and greatest types:

$$\tau ::= \mathsf{bool} \mid \tau_1 \to \tau_2 \mid \{\ell_1 : \tau_1, \ldots, \ell_n : \tau_n\} \mid \bot \mid \top$$

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Types with lattice subtyping:

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What's going on here?

$$\mathcal{F}(A) = \mathsf{Bool}(A) + \mathsf{Func}(A) + \mathsf{Rec}(A)$$

$$\mathsf{Bool}(A) = \mathsf{bool}$$

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Take a lattice of types, forget that it's a lattice, apply $\mathcal{F} : \mathbf{Pos}^{\top}_{\bot} \to \mathbf{Pos}^{\top}_{\bot}$, and notice that the result happens to be a lattice.

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Coproducts

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$$\alpha = (\top \overset{\circ}{\to} \bot) \overset{\circ}{\to} \bot$$

We need type variables as a *free algebra*. Syntactically, add them as opaque indeterminates:

$$\tau ::= \mathsf{bool} \mid \tau_1 \to \tau_2 \mid \{\ell_1 : \tau_1, \ldots, \ell_n : \tau_n\} \mid \alpha$$

Inferring types

Polarity

We have lots of types, but they're not all useful all the time.

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Use *positive types* τ^+ for outputs, and *negative types* τ^- for inputs:

$$\begin{aligned} \tau^+ &::= \alpha \mid \tau^+ \sqcup \tau^+ \mid \bot \mid \mathsf{unit} \mid \tau^- \to \tau^+ \\ \tau^- &::= \alpha \mid \tau^- \sqcap \tau^- \mid \top \mid \mathsf{unit} \mid \tau^+ \to \tau^- \end{aligned}$$

When does ML unify?

Two inputs: introduce $\sqcap$

Two outputs: introduce $\sqcup$

Output used as an input: constraint $\tau^+ \leq \tau^-$

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Handling $\sqcup$ and $\sqcap$ is easy, thanks to polarity:

$$\tau_1 \sqcup \tau_2 \leq \tau_3 \text{ iff } \tau_1 \leq \tau_3, \tau_2 \leq \tau_3$$

$$\tau_1 \leq \tau_2 \sqcap \tau_3 \text{ iff } \tau_1 \leq \tau_2, \tau_1 \leq \tau_3$$

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ML's $\forall$ is more like a set comprehension. These are the same:

$$\forall\alpha\forall\beta.\ \alpha \to \beta \to \alpha$$
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because these are the same

$$\{\alpha \to \beta \to \alpha \mid \alpha, \beta \text{ types}\}$$
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Example

$$\mathsf{filter\_good} : \mathsf{list}[\{\texttt{isgood} : \mathsf{bool}\} \sqcap \alpha] \to \mathsf{list}[\alpha]$$

$$\mathsf{things} : \mathsf{list}[\{\texttt{isgood} : \mathsf{bool}, \texttt{x} : \mathsf{int}\}]$$

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$$\beta^+ \mapsto \beta \sqcup \mathsf{list}[\alpha]$$

$$\alpha^+ \mapsto \alpha \sqcup \{\mathtt{isgood} : \mathsf{bool}, \mathtt{x} : \mathsf{int}\}$$

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Questions?

http://www.cl.cam.ac.uk/~sd601/mlsub

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